

Lower and upper bounds on the mass of light quark-antiquark scalar resonance in the SVZ sum rules

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Abstract

The calculation of the mass of light scalar isosinglet meson within the Shifman–Vainshtein–Zakharov (SVZ) sum rules is revisited. We develop simple analytical methods for estimation of hadron masses in the SVZ approach and try to reveal the origin of their numerical values. The calculations of hadron parameters in the SVZ sum rules are known to be heavily based on a choice of the perturbative threshold. This choice requires some important *ad hoc* information. We show analytically that the scalar mass under consideration has a lower and upper bound which are independent of this choice: $0.78 \lesssim m_s \lesssim 1.28$ GeV.

1 Introduction

The enigmatic σ -meson, called also $f_0(500)$ in the modern Particle Data [1], is attracting a lot of interest for a long time. This lightest scalar resonance emerges usually as an indispensable ingredient in description of the nuclear forces and chiral symmetry breaking in the strong interactions (see, e.g., the recent review [2]). The studies of this meson have a rich and dramatic history [2]. Recently the great efforts have led to a significant progress in reducing the uncertainty in its mass and full width [1]. One observes an increasing evidence in favor of non-ordinary nature of this broad resonance which hardly can be accommodated within the usual quark-antiquark picture of mesons [2].

In the modern literature, the σ -meson is mainly studied in the framework of methods based on analyticity and unitarity [2]. The ensuing models does not have direct relations with QCD, may be except the studies of large- N_c behavior of meson masses. The relations with QCD of many older studies based on effective field theory, bag models, *etc.* [2] are also unclear. Among the phenomenological approaches, the method that perhaps is mostly related to QCD in the spectroscopy of light mesons represents the method of

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Shifman–Vainshtein–Zakharov (SVZ) sum rules [3], often called also ITEP or spectral or just QCD sum rules. The philosophy of this approach is based on the assumption that a quark-antiquark pair (or a more complicated quark current interpolating a hadron) being injected in the strong QCD vacuum does not perturb it noticeably. This allows to parametrize the unknown non-perturbative vacuum by some universal phenomenological characteristics — the vacuum condensates. Hadrons with different quantum numbers have different masses (decay constants, formfactors, *etc.*) because their currents react differently to the vacuum medium. And, roughly speaking, the corresponding coefficients can be calculated from QCD. Assuming further the existence of resonance in some energy range, one is able to calculate its characteristics via the dispersion relations and the Operator Product Expansion (OPE). If one of these assumptions fails, the SVZ method should not work. In practice, this method turned out to be extremely successful in description of hadron parameters [4].

The SVZ sum rules predict that the expected mass of the lightest scalar meson composed of quark and antiquark lies near 1 GeV [5]. In order to obtain a smaller mass, say in the interval 500–700 MeV, one typically needs to consider a four-quark current (see a corresponding review in Ref. [2]). This looks like a confirmation of non-ordinary nature of the σ -meson as long as the assumption of its tetraquark structure works well in various approaches [2].

However, the calculations of hadron masses in the SVZ method involve rather strong assumptions, first of all about a choice of the perturbative threshold. In view of a prominent role that the σ -meson plays both in nuclear and particle physics, it is highly desirable to reduce any *ad hoc* assumptions in studies of this remarkable resonance as much as possible. The main purpose of the present work is to demonstrate explicitly that the mass of the quark-antiquark scalar meson in the SVZ sum rules have a lower bound which is independent of the perturbative threshold and lies above the expected σ -meson mass. As a by-product, we develop a method of simple analytical estimations of hadron masses within the QCD sum rules and demonstrate it in some important cases. In particular, we derive an upper bound on the mass in question.

In numerous papers devoted to the SVZ phenomenology, the values of hadron parameters are customary obtained numerically and demonstrated graphically. The style of our analysis is rather unusual — we will mainly follow various analytical estimates.

The paper is organized as follows. We recall briefly the SVZ method in Section 2. In Section 3, this method is demonstrated in the scalar case. We also explain how the obtained numerical result can be simply calculated analytically. The lower bound on the scalar mass is derived in Section 4. In

Section 5, we obtain the upper bound on this mass. The origin of numerical values of meson masses in the SVZ sum rules is discussed in Section 6. We conclude in Section 7. An application of some of our ideas to the axial-vector case is demonstrated in the Appendix.

2 SVZ method in the scalar sector

We recall briefly the derivation of SVZ sum rules in the scalar sector. Consider the two-point correlation function

$$\Pi_s(p^2) = i \int d^4x e^{ipx} \langle 0 | T \{ j_s(x), j_s(0) \} | 0 \rangle. \quad (1)$$

Here the scalar current is $j_s = \bar{q}q$, where the symbol q stays for the u or d quark. The object (1) contains much dynamical information. In particular, the large-distance asymptotics of (1) in the Euclidean space is $\sim e^{-m_s|x|}$, where m_s is the mass of the ground state, the scalar one in our case. This property lies in the base of the lattice calculations of hadron masses directly from QCD. The central problem in the classical SVZ sum rules consists in the extraction of hadron masses from (1) with the help of some (semi)analytical methods and several phenomenological inputs. The whole approach is based on the two main ideas: The use of the Operator Product Expansion for (1) and a representation of the correlator (1) via a suitable dispersion relation.

In the Euclidean domain ($p^2 = -Q^2$), the OPE for (1) reads [5]

$$\begin{aligned} \Pi_s(Q^2) = \frac{3}{16\pi^2} \left(1 + \frac{11}{3} \frac{\alpha_s}{\pi} \right) Q^2 \log \frac{Q^2}{\mu^2} + \frac{\alpha_s}{16\pi} \frac{\langle G^2 \rangle}{Q^2} + \frac{3}{2} \frac{m_q \langle \bar{q}q \rangle}{Q^2} \\ - \frac{88}{27} \pi \alpha_s \frac{\langle \bar{q}q \rangle^2}{Q^4} + \mathcal{O} \left(\frac{1}{Q^6} \right), \quad (2) \end{aligned}$$

where $\langle G^2 \rangle$ and $\langle \bar{q}q \rangle$ denote the gluon and quark vacuum condensate, respectively. In the practical calculations of masses for the light non-strange mesons, one neglects: (i) The running of α_s and of the factor in front of α_s in the unit operator; (ii) The $\mathcal{O}(m_q^2)$ and $\mathcal{O}(1/Q^6)$ contributions; (iii) A small anomalous dimension of $\alpha_s \langle \bar{q}q \rangle^2$. The vacuum saturation hypothesis [3] is exploited for the dimension-six operator in (2) (i.e. the factorization $\langle \bar{q}\Gamma q \bar{q}\Gamma q \rangle \sim [(\text{Tr}\Gamma)^2 - (\text{Tr}\Gamma^2)] \langle \bar{q}q \rangle^2$ which can be justified in the large- N_c limit of QCD) and the value of $\alpha_s \langle \bar{q}q \rangle^2$ absorbs small contribution from other dimension-six operators ($m_q \langle \bar{q}Gq \rangle$ and $\langle G^3 \rangle$).

On the other hand, the correlation function (1) satisfies the twice-subtracted

dispersion relation,

$$\frac{d^2\Pi_s(p^2)}{d(p^2)^2} = \frac{2}{\pi} \int_{4m_q^2}^{\infty} ds \frac{\text{Im}\Pi_s(s)}{(s-p^2)^3}. \quad (3)$$

In the SVZ sum rules, one usually assumes the so-called "one resonance plus continuum" ansatz for the spectral function,

$$\text{Im}\Pi_s = f_s^2 s \delta(s - m_s^2) + \frac{3}{16\pi^2} \left(1 + \frac{11}{3} \frac{\alpha_s}{\pi}\right) s \Theta(s - s_0). \quad (4)$$

Here the scalar "decay constant" is defined by the matrix element of scalar current between the vacuum and a scalar state f_0 ,

$$\langle 0 | j_s | f_0 \rangle = f_s m_s. \quad (5)$$

The ansatz (4) does not take into account the decay width, i.e. the resonance is considered as infinitely narrow. The higher resonances and various thresholds in the spectral function are absorbed into the perturbative continuum.

In order to suppress the higher dimensional condensates in the OPE and simultaneously enhance the relative contribution of the ground state into the correlator, the SVZ method makes use of the Borel transform,

$$L_M \Pi(Q^2) = \lim_{\substack{Q^2, n \rightarrow \infty \\ Q^2/n = M^2}} \frac{1}{(n-1)!} (Q^2)^n \left(\frac{-d}{dQ^2} \right)^n \Pi(Q^2). \quad (6)$$

The transform (6) suppresses operators of dimension $2k$ by a factor of $1/(k-1)!$ and provides an exponential suppression of the kind of $e^{-m_n^2/M^2}$ for heavier resonances ("radial excitations" with masses m_n) in the spectral function (4). Applying (6) to (twice derivative of) (2) and (3) with ansatz (4) one arrives at the expression $f_s^2 e^{-m_s^2/M^2} = \dots$. Another relation can be obtained by considering the derivative $\frac{d}{d(1/M^2)} (-f_s^2 e^{-m_s^2/M^2}) = f_s^2 m_s^2 e^{-m_s^2/M^2} = \dots$ (the ensuing relation represents the sum rule for derivative of $Q^2 \Pi_s(Q^2)$). Dividing the second relation by the first one, we get finally [5]

$$m_s^2 = M^2 \frac{2h_0 \left[1 - \left(1 + \frac{s_0}{M^2} + \frac{s_0^2}{2M^4} \right) e^{-s_0/M^2} \right] + \frac{h_3}{M^6}}{h_0 \left[1 - \left(1 + \frac{s_0}{M^2} \right) e^{-s_0/M^2} \right] + \frac{h_2}{M^4} - \frac{h_3}{M^6}}, \quad (7)$$

where

$$h_0 = 1 + \frac{11}{3} \frac{\alpha_s}{\pi}, \quad (8)$$

$$h_2 = \frac{\pi^2}{3} \left(\frac{\alpha_s}{\pi} \langle G^2 \rangle + 24m_q \langle \bar{q}q \rangle \right), \quad (9)$$

$$h_3 = \frac{1408}{81} \pi^3 \alpha_s \langle \bar{q}q \rangle^2. \quad (10)$$

The numerical values taken in Ref. [5] are: $\alpha_s = 0.6$, $h_2 = 0.04 \text{ GeV}^4$, $h_3 = 0.08 \text{ GeV}^6$.

3 Calculation of the scalar mass

The extraction of scalar mass from the expression (7) is a matter of art as it requires some additional information. First, the result is sensitive to the choice of the continuum threshold s_0 . So *a priori* we need to guess the mass region where our resonance should lie and we should have certain ideas on the mass of the next scalar meson (the first "radial excitation") and on positions of various thresholds in the spectral function. Second, the prediction should be stable against variations of the borel parameter M in the so-called "Borel window" $a < M^2 < b$ in which we hope to find our resonance. In the region $M^2 < a$, the power corrections in OPE (2) become important and we cannot consider the first few terms only. In the region $M^2 > b$, the heavier resonances and various thresholds contribute to the spectral function (4). After some discussions, the choice made in Ref. [5] was (in GeV^2) $s_0 \simeq 1.5$ and $0.8 < M^2 < 1.4$ that resulted in the prediction

$$m_s = 1.00 \pm 0.03 \text{ GeV}. \quad (11)$$

The given estimate is usually regarded as a standard prediction for the scalar mass in the SVZ sum rules.

We wish to make a comment on this result which will be important in what follows. In the classical case of ρ -meson [3], the mass m_ρ^2 as a function of M^2 has a rather deep minimum. The Borel window lies around this minimum near $M^2 \simeq m_\rho^2$ and the dependence of m_ρ^2 on s_0 is not strong. The scalar case is different: The minimum of function $m_s^2(M^2)$ in (7) is very shallow and hardly visible in the practical calculations. In this case, the dependence on s_0 becomes much stronger. Our key observation is that within the accuracy displayed in (11), the value of m_s^2 predicted in the Borel window practically coincides with its asymptotic value at $M^2 \rightarrow \infty$,

$$m_{\text{asympt}}^2 = \frac{\frac{1}{3}h_0s_0^3 + h_3}{\frac{1}{2}h_0s_0^2 + h_2}. \quad (12)$$

The asymptotics (12) is independent of $\mathcal{O}(1/M^8)$ condensate contributions omitted in (7) but deviations of (12) from m_s^2 calculated in the Borel window do depend on these contributions. If we accept the accuracy 0.03 GeV as in (11), we will have $m_{\text{asympt}} - m_s \lesssim 0.03 \text{ GeV}$ when $s_0 \lesssim 1.7 \text{ GeV}^2$. For $s_0 \simeq 1.5 \text{ GeV}^2$ used in (11), the deviation is $m_{\text{asympt}} - m_s \simeq 0.01 \text{ GeV}$.

If s_0 in (12) is large enough the condensate contributions h_2 and h_3 are not substantial. In this "physical" case, we obtain from (12) a simple expression for the scalar mass,

$$m_s^2 \simeq \frac{2}{3}s_0. \quad (13)$$

Taking the input $s_0 \simeq 1.5 \text{ GeV}^2$ used for the numerical result (11), this result follows immediately. The relation (13) also explains why (as was observed in Ref. [5] numerically) m_s is very insensitive to a change in α_s : The dependence on α_s appears only in the next-to-leading level in the $\frac{1}{s_0}$ expansion, $m_s^2 \simeq \frac{2}{3}s_0 - \frac{4h_2}{3h_0s_0}$.

To justify further the usefulness of the limit $M^2 \rightarrow \infty$, in Appendix, we demonstrate how it works in the axial-vector sector.

4 Lower bound on the scalar mass

The scalar mass m_s depends rather strongly on the choice of perturbative cutoff parameter s_0 , however, there is a lower bound on the value of m_s . Since in the region $s_0 \lesssim 1.5 \text{ GeV}^2$ the asymptotic formula (12) yields a result practically indistinguishable from the exact numerical value within the accuracy of the SVZ sum rules, we first consider this region.

This bound can be obtained from calculation of minimum of $m_s^2(s_0)$ from (12). We have

$$\frac{dm_s^2}{ds_0} = \frac{h_0s_0}{\left(\frac{1}{2}h_0s_0^2 + h_2\right)^2} \left[s_0 \left(\frac{1}{2}h_0s_0^2 + h_2 \right) - \left(\frac{1}{3}h_0s_0^3 + h_3 \right) \right] = 0. \quad (14)$$

The equation (14) is quartic in s_0 . It has 4 solutions and $s_0 = 0$ is one of them. Thus, there exists at least another one real solution that we will denote \bar{s}_0 . In fact, at positive h_2 and h_3 , the remaining two solutions are always complex since the imaginary part of roots is $\pm i\frac{\sqrt{3}}{2} \left(H + \frac{2h_2}{h_0H} \right)$, where H is given by (20). It is easy to check that

$$\left. \frac{d^2m_s^2}{ds_0^2} \right|_{s_0=0} = -\frac{h_0h_3}{h_2^2} < 0. \quad (15)$$

Hence, $s_0 = 0$ delivers the local maximum to (12). Thus, $s_0 = \bar{s}_0$ corresponds to the (global at $s_0 > 0$) minimum of (12). To calculate $m_s^2(\bar{s}_0)$ we do not need the exact expression for \bar{s}_0 . Equating to zero the square brackets in (14) we get

$$\bar{s}_0 = \frac{\frac{1}{3}h_0\bar{s}_0^3 + h_3}{\frac{1}{2}h_0\bar{s}_0^2 + h_2}. \quad (16)$$

Comparing (16) with (12) we conclude immediately that

$$m_s^2|_{\min} = \bar{s}_0, \quad (17)$$

where, from (16), \bar{s}_0 is the real solution of the cubic equation

$$\frac{1}{6}h_0s_0^3 + h_2s_0 - h_3 = 0. \quad (18)$$

This solution can be written explicitly,

$$\bar{s}_0 = H - \frac{2h_2}{h_0H}, \quad (19)$$

$$H = \left(\frac{\sqrt{9h_3^2 + 8\frac{h_2^3}{h_0}} + 3h_3}{h_0} \right)^{1/3}. \quad (20)$$

For further numerical estimates, we explain in detail how the input parameters determining h_0 , h_2 and h_3 in (8)–(10) are fixed. The GOR relation for the pion mass, $m_\pi^2 f_\pi^2 = -(m_u + m_d)\langle\bar{q}q\rangle$ can be directly derived in the SVZ sum rules [3]. This allows to fix the renorminvariant dim-4 condensate $m_q\langle\bar{q}q\rangle = -\frac{1}{2}m_\pi^2 f_\pi^2$, where $m_\pi = 140$ MeV and $f_\pi = 92.4$ MeV [1]. To fix $\alpha_s\langle\bar{q}q\rangle^2$ we will exploit the approximate renorminvariance of this dim-6 condensate. Since $\alpha_s|_{\mu=2\text{GeV}} \simeq 0.3$ and $(m_u + m_d)|_{\mu=2\text{GeV}} \simeq 7$ MeV [1], we have $\alpha_s\langle\bar{q}q\rangle^2 \simeq 0.3\frac{m_\pi^4 f_\pi^4}{0.007^2}$ GeV⁶. We will use α_s at the scale 1 GeV where the scalar meson is expected. As $m_q|_{\mu=1\text{GeV}} \simeq 1.35 m_q|_{\mu=2\text{GeV}}$ [1], we get $\langle\bar{q}q\rangle|_{\mu=1\text{GeV}} \simeq 1.35^{-1} \langle\bar{q}q\rangle|_{\mu=2\text{GeV}}$ and the approximate renorminvariance of $\alpha_s\langle\bar{q}q\rangle^2$ leads to $\alpha_s|_{\mu=1\text{GeV}} \simeq 1.35^2 \alpha_s|_{\mu=2\text{GeV}} \simeq 0.55$. The mostly used phenomenological value of the gluon condensate is $\frac{\alpha_s}{\pi}\langle G^2\rangle \simeq (0.36 \text{ GeV})^4$.

With the input values above, the relations (8)–(10) fix $h_0 \simeq 1.64$, $h_2 \simeq 0.049$ GeV⁴, $h_3 \simeq 0.092$ GeV⁶. Using these inputs, the relations (17)–(20) yield the following estimate: $m_s|_{\min} \simeq 0.78$ GeV. This lower bound practically coincides with the ω -meson mass [1]. Taking α_s at the scale of the ω -meson mass, $\alpha_s \simeq 0.7$ (as it was originally used in Ref. [3]), one gets a bit corrected estimate: $m_s|_{\min} \simeq 0.77$ GeV. The uncertainty in determination of the gluon condensate, $\frac{\alpha_s}{\pi}\langle G^2\rangle \simeq (0.36 \pm 0.02 \text{ GeV})^4$, leads to the uncertainty $m_s|_{\min} \simeq 0.77 \pm 0.01$ GeV.

A further correction may come from a more accurate treatment of the large- N_c limit in QCD [6, 7]. Since this limit was exploited to justify both the narrow-width approximation in the spectral function (4) and the vacuum saturation hypothesis for the dim-6 condensate in the OPE (2), it looks more consistent to use the large- N_c limit also for the factor in front of the dim-6

operator in the OPE (2): $\frac{88}{27} = \frac{11}{9} \left(N_c - \frac{1}{N_c} \right)$. This point is usually ignored in the SVZ phenomenology. Neglecting the $1/N_c$ term, i.e. replacing $\frac{88}{27} \rightarrow \frac{11}{3}$, slightly shifts up our estimate: $m_s|_{\min} \simeq 0.77 \rightarrow 0.79$ GeV.

In summary, we would estimate the lower bound of the scalar mass as

$$m_s|_{\min} = 0.78 \pm 0.02 \text{ GeV}. \quad (21)$$

It is seen that at physical values of condensates the lower bound for the mass of scalar meson lies around the mass of the ω -meson. If we imposed by hands a typical σ -meson mass, say $m_\sigma \simeq 0.45$ GeV [1, 2], as the lower bound at physical $\langle \bar{q}q \rangle$, we would obtain that this is possible at a huge value of the gluon condensate, $\frac{\alpha_s}{\pi} \langle G^2 \rangle \simeq (0.61 \text{ GeV})^4$. Alternatively, we could fix the physical value for $\frac{\alpha_s}{\pi} \langle G^2 \rangle$ and obtain then a rather small quark condensate $\langle \bar{q}q \rangle|_{\mu=1\text{GeV}} \simeq -(0.19 \text{ GeV})^3$ (its physical value lies around $-(0.25 \text{ MeV})^3$ at $\mu = 1 \text{ GeV}$).

Consider now the region $s_0 \gtrsim 1.5 \text{ GeV}^2$. Here the deviation of calculated mass from the asymptotics (12) becomes visible. However, the predicted value of the scalar mass exceeds the lower bound (21) significantly, $m_s \gtrsim 1 \text{ GeV}$, and grows with s_0 . Thus, this region does not change our estimate (21).

5 Upper bound on the scalar mass

If the perturbative threshold s_0 is high enough, say $s_0 \gtrsim 2 \text{ GeV}^2$, the minimum of the function $m_s^2(M^2)$ in (7) becomes prominent. The Borel window is situated around this minimum. When s_0 grows, the competition between the perturbative continuum and power corrections narrows the Borel window. In the limit $s_0 \rightarrow \infty$, the Borel window shrinks to one "fixed point" M_0 which represents the minimum of $m_s^2(M^2)$. The scalar mass, as a quantity calculated in the Borel window, grows with s_0 and reaches its maximal value in this point, $m_s|_{\max} = m_s(M_0^2)$.

Regarding the limit $s_0 \rightarrow \infty$, one may wonder why we may neglect the contribution of thresholds and higher resonances to the spectral function (4)? First of all, the ansatz (4) is rough as the decay width is neglected. The narrow-width approximation is justified in the large- N_c limit of QCD [6, 7]. All thresholds related with particle decays are suppressed in this limit and should be then also neglected. If an analysis is based on the large- N_c arguments, one cannot pretend to the accuracy better than, say, 10%. In practice, the contribution of heavier resonances (the radial excitations) to the spectral function is rather small and comparable with 10%. Thus a

possible error from omitting the radial states lies within the accuracy of the method itself.

With this caveat in mind, consider the limit $s_0 \rightarrow \infty$ in (7),

$$m_{s_0 \rightarrow \infty}^2 = M^2 \frac{2h_0 + \frac{h_3}{M^6}}{h_0 + \frac{h_2}{M^4} - \frac{h_3}{M^6}}. \quad (22)$$

The minimum of expression (22) follows from the condition $\frac{dm_{s_0 \rightarrow \infty}^2}{dM^2} = 0$ that leads to the polynomial equation

$$2h_0^2(M^2)^6 + 6h_0h_2(M^2)^4 - 10h_0h_3(M^2)^3 - h_3^2 = 0. \quad (23)$$

With our numerical inputs above, the positive real solution of (23) is

$$M_0 \simeq 0.78 \text{ GeV}. \quad (24)$$

Substituting this value to (22) we conclude that the calculation of the scalar mass in the Borel window cannot exceed the upper limit

$$m_s|_{\text{max}} \simeq 1.28 \text{ GeV}. \quad (25)$$

The upper bound (25) practically coincides with the mass of $f_1(1285)$ -meson [1] which, like a usual scalar meson, is also a P -wave quark-antiquark state.

6 Discussions

It is interesting to observe a numerical coincidence of the Borel "fixed point" M_0 in (24) and the lower bound for the scalar mass (21). This coincidence is not completely accidental. The value of h_3^2 in the Eq. (23) is numerically very small, omitting this term does not affect the result (24) within our accuracy. The Eq. (23) is then cubic in M^2 and can be solved analytically. The most important point here is that the relative contribution of the term with h_2 into the real solution of both Eq. (23) and Eq. (18) is relatively small (numerically by almost two orders of magnitude) in comparison with the term containing h_3 . This allows to write immediately the approximate real solution of Eq. (18), $\sqrt{s_0} \simeq (6h_3/h_0)^{1/6}$, and of Eq. (23), $M_0 \simeq (5h_3/h_0)^{1/6}$. the relative difference is $(6/5)^{1/6} \simeq 1.03$ and explains qualitatively the numerical coincidence of M_0 with $m_s|_{\text{min}}$. In addition, the given observation demonstrates that both $m_s|_{\text{min}}$ and (via M_0) $m_s|_{\text{max}}$ are determined mainly by the value of the condensate of the highest dimension kept in the OPE.

Physically M_0 can be interpreted as the parameter determining the left border of the Borel window. When the perturbative threshold is moved from

infinity to some finite physical value, say around $s_0 \simeq 1.5 \text{ GeV}^2$, the left border slightly shifts right (enlarges) from M_0 . As a consequence, the value of mass calculated in the Borel window slightly moves down from the maximal bound, in the scalar case — the expression (22). Here the term "slightly" means "within 10–15%", i.e. within the accuracy of the SVZ method. The seemingly strong dependence of mass on s_0 , expressed by the relation (13), appears only in a very limited interval of s_0 . In reality, however, the extracted mass is mainly determined by the value of M_0 . Indeed, this value has an intermediate position between the physical cutoff $\sqrt{s_0}$ and (a proper power of) condensate contributions. This allows to write a simple estimate for the scalar mass just neglecting the exponential and power terms in (7) (they are suppressed simultaneously),

$$m_s^2 \simeq 2M_0^2. \quad (26)$$

This estimate yields $m_s \simeq 1.1 \text{ GeV}$ that agrees within 10% with (11).

Consider the classical case of ρ -meson. The mass relation is [3, 5]

$$m_\rho^2 = M^2 \frac{h_0 \left[1 - \left(1 + \frac{s_0}{M^2} \right) e^{-s_0/M^2} \right] - \frac{h_2}{M^4} - \frac{h_3}{M^6}}{h_0 [1 - e^{-s_0/M^2}] + \frac{h_2}{M^4} + \frac{h_3}{2M^6}}, \quad (27)$$

where $h_0 = 1 + \frac{\alpha_s}{\pi}$ with $\alpha_s = 0.7$, $h_2 = 0.046 \text{ GeV}^4$, $h_3 \simeq -0.064 \text{ GeV}^6$. Following the same consideration, we would find $M_0 \simeq 0.72 \text{ GeV}$ and

$$m_\rho \simeq M_0. \quad (28)$$

The given estimate justifies the assumption $m_\rho \simeq M$ used in the original paper [3], where M was an "optimal" region in the Borel window.

In the classical SVZ sum rules, one usually estimates the contribution of condensates to the masses on the level of 5–10% [3]. We can trace qualitatively how this estimate originates within our approach: Roughly speaking, the estimate stems from neglecting the power corrections in (28) or (26). However, we can directly see that this estimate is not well justified because the value of M_0 is completely determined by the condensates, as was shown above. To put it differently, the scale of a meson mass extracted from the SVZ sum rules is mainly dictated not by the perturbative threshold s_0 (as it might seem) but by the scale of the Borel parameter M in the Borel window. And this latter is determined (via M_0) by the condensates!

Recalling that the numerical value of M_0 comes mainly from the condensate of the highest dimension kept in the OPE, it looks like a miracle that the SVZ sum rules work well neglecting all higher power corrections. A partial reason might be the fact that the values of condensates in the SVZ method are normalized to the phenomenology and an account for higher

power terms could lead to a double counting of non-perturbative effects. But the main reason of phenomenological success of the SVZ method seems to consist in a fortunate parametrization of important universal properties of the non-perturbative QCD vacuum.

It should be emphasized that we have used the simplest SVZ sum rules framework, i.e. we employed in the scalar sector the same set of assumptions as in the vector one. Our wish was to escape any additional model assumptions and related complications. In reality, the scalar sector is known to be much more complicated than the vector one. A more advanced study of the correlator of the scalar currents should include, e.g. higher order perturbative contributions, instanton contributions, finite width effects, and mixing with glueballs. An example of such an analysis for the non-strange quark-antiquark scalar sector is given in Refs. [8,9].

7 Conclusions

Within the framework of SVZ sum rules, we have shown that the mass of scalar resonance interpolated by the quark-antiquark current has a lower and upper bound, $0.78 \lesssim m_s \lesssim 1.28$ GeV. These bounds do not depend on the choice of the perturbative threshold. The values of bounds are determined by the dim-4 and dim-6 condensates in the OPE, with the main contribution stemming from the dim-6 one.

Our analysis confirms a widespread idea that the $f_0(500)$ -meson represents an exotic state. The scalar isoscalar state described by the SVZ method can physically correspond to $f_0(980)$ or $f_0(1370)$. The latter possibility is much less likely as the mass uncertainty of $f_0(1370)$, 1200–1500 MeV [1], has a relatively small overlap with the upper bound.

Our estimations can be extended to the scalar strange current $j = \bar{s}s$. The bounds are expected to shift up by several hundreds MeV. The (poorly known) higher dimensional condensates should affect our analysis. The account for their effects is an open problem.

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Appendix

The limit $M^2 \rightarrow \infty$ is useful in the cases where a hadron mass as a function of the Borel parameter M does not have a prominent extremum. In the given Appendix, we show how the axial-vector sector is simplified in this limit.

Since the axial-vector current $j_\mu^a = \bar{q}\gamma_\mu\gamma_5 q$ is not conserved, the two-point correlator (1) of these currents has (in contrast to the vector case) two independent contributions,

$$\Pi_{\mu\nu}^a(p^2) = -\Pi_1(p^2)g_{\mu\nu} + \Pi_2(p^2)p_\mu p_\nu. \quad (29)$$

The sum rules for Π_1 and Π_2 are different with different final expressions for the axial-vector mass. The case of Π_1 is similar to the scalar one and the resulting mass formula is in one-to-one correspondence with (7) [5],

$$m_a^2 = M^2 \frac{2h_0 \left[1 - \left(1 + \frac{s_0}{M^2} + \frac{s_0^2}{2M^4} \right) e^{-s_0/M^2} \right] + \frac{h_3}{M^6}}{h_0 \left[1 - \left(1 + \frac{s_0}{M^2} \right) e^{-s_0/M^2} \right] - \frac{h_2}{M^4} - \frac{h_3}{M^6}}. \quad (30)$$

We will use the numerical values of parameters from Ref. [5]: $h_0 = 1 + \frac{\alpha_s}{\pi}$ with $\alpha_s = 0.6$, $h_2 = 0.046 \text{ GeV}^4$, $h_3 \simeq 0.10 \text{ GeV}^6$. The longitudinal part Π_2 contains a contribution from the pion pole which enters the expression for the mass via the power term with the factor $h_1 = 8\pi^2 f_\pi^2 \simeq 0.67 \text{ GeV}^2$ [3, 5],

$$m_a^2 = M^2 \frac{2h_0 \left[1 - \left(1 + \frac{s_0}{M^2} \right) e^{-s_0/M^2} \right] - \frac{h_2}{M^4} - \frac{h_3}{M^6}}{h_0 \left[1 - e^{-s_0/M^2} \right] - \frac{h_1}{M^2} + \frac{h_2}{M^4} + \frac{h_3}{2M^6}}. \quad (31)$$

The numerical calculations show that the expressions (30) and (31) result in the same mass for $s_0 \simeq 1.75 \text{ GeV}^2$ which gives the mass

$$m_a = 1.15 \pm 0.04 \text{ GeV}. \quad (32)$$

As in the relation (26), we can write a simple estimate $m_a^2 \simeq 2M_0^2$. Here the value of M_0 is determined by the Eq. (23) with the opposite sign for the term containing h_2 . The solution is $M_0 \simeq 0.9 \text{ GeV}$, where the numerical difference with (24) comes mainly from a different h_0 . We have thus the estimate $m_a \simeq 1.27 \text{ GeV}$ that agrees within 10% with (32).

Consider now the limit $M^2 \rightarrow \infty$ in the expressions (30) and (31). We get the lower asymptotics

$$m_{\text{asympt}}^2 = \frac{\frac{1}{3}h_0 s_0^3 + h_3}{\frac{1}{2}h_0 s_0^2 - h_2}, \quad (33)$$

for (30) and the upper asymptotics

$$m_{\text{asyp}}^2 = \frac{\frac{1}{2}h_0s_0^2 - h_2}{h_0s_0 - h_1}, \quad (34)$$

for (31). Equating (33) with (34), one obtains a quartic polynomial equation for s_0 which has analytical solutions. The result is $s_0 \simeq 1.79 \text{ GeV}^2$ and $m_a \simeq 1.13 \text{ GeV}$. It is seen that the difference with the exact numerical solution of Ref. [5] is small.

As in the scalar case, we can try to neglect the contribution of condensates in (33) and get a simple approximate relation (13) for the axial mass,

$$m_a^2 \simeq \frac{2}{3}s_0. \quad (35)$$

This would lead to the estimate $m_a \simeq 1.09 \text{ GeV}$ differing from the numerical solution by less than 10%.

In the asymptotical expression (34), the limit $s_0 \rightarrow \infty$ for the mass estimation is bad since the contribution from h_1 is not relatively small. Let us expand (34) in $\frac{1}{s_0}$ up to the next-to-leading term,

$$m_a^2 \simeq \frac{1}{2}s_0 + \frac{h_1}{2h_0}. \quad (36)$$

Equating (35) with (36) we arrive at the estimate

$$s_0 \simeq \frac{3h_1}{h_0} = \frac{24\pi^2 f_\pi^2}{1 + \alpha_s/\pi} \simeq 1.7 \text{ GeV}^2, \quad (37)$$

which is very close to the numerical value used in (32). Substituting (37) in (35), we can express the axial-vector mass via f_π ,

$$m_a^2 \simeq \frac{16\pi^2 f_\pi^2}{1 + \alpha_s/\pi}. \quad (38)$$

As was advocated in the original Ref. [3] on the SVZ sum rules, saturation of spectral function $\frac{1}{\pi}\text{Im}\Pi_2$ by only the pion pole below the ρ -meson mass leads to a successful relation $m_\rho^2 \simeq \frac{8\pi^2 f_\pi^2}{1 + \alpha_s/\pi}$. Combining this relation with (38) we re-derive the famous Weinberg relation [10], $m_a^2 \simeq 2m_\rho^2$.

References

- [1] K. A. Olive *et al.* (Particle Data Group), Chin. Phys. **C 38**, 090001 (2014).

- [2] J. R. Pelaez, arXiv:1510.00653 [hep-ph].
- [3] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B **147**, 385 (1979).
- [4] P. Colangelo and A. Khodjamirian, hep-ph/0010175.
- [5] L.J. Reinders, H. Rubinstein, S. Yazaki, *Phys. Rept.* **127**, 1 (1985).
- [6] G. 't Hooft, Nucl. Phys. B **72**, 461 (1974).
- [7] E. Witten, Nucl. Phys. B **160**, 57 (1979).
- [8] V. Elias, A. H. Fariborz, F. Shi and T. G. Steele, Nucl. Phys. A **633**, 279 (1998).
- [9] D. Harnett, R. T. Kleiv, K. Moats and T. G. Steele, Nucl. Phys. A **850**, 110 (2011).
- [10] S. Weinberg, Phys. Rev. Lett. **18**, 507 (1967).